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Section 12.5 : Lines & Planes

a line in 3-space is given by parametrized vector equation $\boxed{l(t) = \vec{p} + t\vec{v}}$ where $\vec{p}$ = position vector of a point on l and $\vec{v}$ = direction of line

Ex: Compute the vector equation of the line through $(-6, 2, 3)$ and parallel to line

$$m(t) = \langle 0, 2, -1 \rangle + t\langle -2, 1, 5 \rangle$$

Sol

Given: $\vec{p} = \langle -6, 2, 3 \rangle$. Because of parallelism, $\vec{v} = \langle -2, 1, 5 \rangle$ is a valid direction vector.

$\therefore$ $l(t) = \langle -6, 2, 3 \rangle + t\langle -2, 1, 5 \rangle$

The parametric equations of a line are $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ which are the component functions of the vector form

Ex: For l and m as in the previous example, simplify the vector equation:

$l(t) = (-6, -2t, 2+t, 3+5t)$ $\quad \begin{cases} x = -6-2t \\ y = 2+t \\ z = 3+5t \end{cases}$

$m(t) = (-2t, 2+t, -1+5t)$ $\quad \begin{cases} x = -2t \\ y = 2+t \\ z = -1+5t \end{cases}$

$\therefore$ has parametric equations

A line can also be represented (sometimes) by symetric equations

(solved for parameter) $\rightsquigarrow \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$

Ex: For l as above, has parametric equations:

$\begin{cases} x = -6-2t \\ y = 2+t \\ z = 3+5t \end{cases} \rightsquigarrow \begin{cases} \frac{x+6}{-2} = t \\ \frac{y-2}{1} = t \\ \frac{z-3}{5} = t \end{cases} \rightsquigarrow \boxed{\frac{x+6}{-2} = \frac{y-2}{1} = \frac{z-3}{5}}$

Symetric equations of l

This procedure can be done in reverse too; if given symetry equations we can easily get parametric equations too

Some Terminology: Two lines are ...

① parallel if their direction vectors are parallel
② intersecting if they have a point in common
③ skew if they are neither parallel or intersecting

Ex: Classify as parallel, intersecting, or skew:

$l_1(t) = \langle 5-12t, 3+9t, 1-3t \rangle$
$l_2(t) = \langle 3+8t, -6t, 7+2t \rangle$

sol: $l_1(t) = \langle 5,3,1 \rangle + t\langle -12, 9, -3 \rangle$
$l_2(t) = \langle 3,0,7 \rangle + t\langle 8, -6, 2 \rangle$
] not equal, but we must check the unit vectors

unit vectors:

$\frac{1}{|\vec{v}_1|}\vec{v}_1 = \frac{1}{\sqrt{234}}\langle -12, 9, -3 \rangle = \frac{1}{3\sqrt{26}}\langle -12, 9, -3 \rangle = \frac{1}{\sqrt{26}}\langle -4, 3, -1 \rangle$

$\frac{1}{|\vec{v}_2|}\vec{v}_2 = \frac{1}{\sqrt{109}}\langle 8, -6, 2 \rangle = \frac{1}{2\sqrt{26}}\langle 8, -6, 2 \rangle = \frac{1}{2\sqrt{26}}\langle 4, -3, 1 \rangle$

Notice: $\frac{1}{|\vec{v}_2|}\vec{v}_2 = -\frac{1}{|\vec{v}_1|}\vec{v}_1$, so l_1 is parallel to l_2

check if they're intersecting:
DO NOT solve $l_1(t_0) = l_2(t_0)$. Instead, solve for something like $l_1(t) = l_2(s)$ => do paths cross, not necessarily at the same time:

$\langle 5-12t, 3+9t, 1-3t \rangle = \langle 3+8s, -6s, 7+2s \rangle$

i.e. $\begin{cases} 5-12t = 3+8s \\ 3+9t = -6s \\ 1-3t = 7+2s \end{cases} \Rightarrow \begin{cases} -12t-8s = -2 \\ 9t+6s = -3 \\ -3t-2s = 6 \end{cases}$ => This is a system of equations

Ex Cont: $\begin{cases} 6t+4s=1 \\ 3t+2s=-1 \\ 3t+2s=-6 \end{cases}$ $\Rightarrow$ implies $-1 = 3t+2s = -6$ which is simply not a valid expression.

$\therefore$ the lines are not intersecting

Recall: A plane in 3-space has vector equation

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

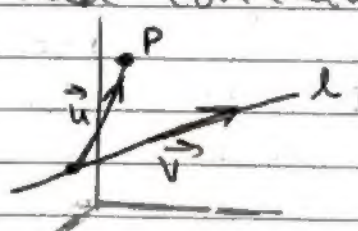
normal vector; vector of variables; any position point on the plane

Ex: Compute the plane through $(1,2,4)$ and perpendicular to $\langle -2,1,3 \rangle$:

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \Rightarrow \langle -2,1,3 \rangle \cdot \langle x-1, y-2, z-4 \rangle$$

$$= -2(x-1) + 1(y-2) + 3(z-4) = 0$$

Ex: Compute the plane through the point $(3,5,-1)$ and containing the line $\begin{cases} x = 4-t \\ y = 2t-1 \\ z = -3t \end{cases}$



Sol: $\vec{p} = (3,5,-1)$

need Q, a point on l $\Rightarrow$ pick a time

so lets use $Q = l(0) = (4,-1,0)$

So, $\vec{n} = \langle 3-4, 5--1, -1-0 \rangle = \langle -1,6,-1 \rangle$

$l(t) = \langle 4,-1,0 \rangle + t\underbrace{\langle -1,0,-3 \rangle}_{\vec{v}}$

Ex Cont. $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 6 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \langle -16, -2, 4 \rangle$

$\therefore$ the plane has equation $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$

i.e. $\langle -16, -2, 4 \rangle \cdot \langle x-3, y-5, z+1 \rangle = 0$

$\Rightarrow -16(x-3) - 2(y-5) + 4(z+1) = 0$

Section 12.6: Quadratic Surfaces

IDEA: We want to study degree 2 polynomials and solution sets in 3-space.

Ex: $p(x,y,z) = x^2 - z$ ← "degenerate" b/c it doesn't depend on ALL variables

solution set: $p(x,y,z) = 0$ iff $x^2 - z = 0$

$\Rightarrow x^2 = z$

In the xz plane, this looks like a parabola

This solution set is actually a (parabolic) cylinder

A picture in 3-space: